Kinematic Limit Analysis of Nonassociated Perfectly Plastic Material by the Bipotential Approach and Finite Element Method

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Limit analysis is one of the most fundamental methods of plasticity. For the nonstandard model, the concept of the bipotential, representing the dissipated plastic power, allowed us to extend limit analysis theorems to the nonassociated flow rules. In this work, the kinematic approach is used to find the limit load and its corresponding collapse mechanism. Because the bipotential contains in its expression the stress field of the limit state, the kinematic approach is coupled with the static one. For this reason, a solution of kinematic problem is obtained in two steps. In the first one, the stress field is assumed to be constant and a velocity field is computed by the use of the kinematic theorem. Then, the second step consists to compute the stress field by means of constitutive relations keeping the velocity field constant and equal to that of the previous step. A regularization method is used to overcome problems related to the nondifferentiability of the dissipation function. A successive approximation algorithm is used to treat the coupling question. A simple compression-traction of a nonassociated rigid perfectly plastic material and an application of punching by finite element method are presented in the end of the paper.
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1 Introduction

In solid mechanics, the knowledge of the plastic limit state is very important in the design of structures according to a plastic strength criterion. To deal with, the modern limit analysis is a direct and an efficient tool. In the frame of the standard model of the plasticity, the modern limit analysis is founded on rigorous mathematical formulation and serious computational methods as convex analysis and mathematical programming [1–11]. However, in the case of the nonassociated plasticity, constitutive laws usually proposed in the literature require the use of two stress functions, the yield function and the so-called plastic potential [12]. Using this model, the limit analysis cannot be applied.

On the other hand, an alternative approach to the nonassociated plasticity stems from the model of implicit standard materials. The crucial idea consists of introducing the so-called bipotential, depending on both the stress and plastic strain rate. The knowledge of this unique function allows simultaneously defining the yield locus and the yielding rule. In the frame of the convex analysis, the bipotential approach extends in a natural way Fenchel's inequality [13] and Moreau's superpotential [14]. This idea was successfully applied to Coulomb's dry friction [15–17] and the soil materials [18–20]. Recently, a straightforward extension of the limit analysis bound theorems was proposed for the nonassociated flow rules by De Saxcé and Bousshine [21]. This extension allows limit analysis to be applied to more real engineering problems.

The kinematic approach consists of minimizing the plastic dissipation power, represented here by the plastic bipotential, throughout the rigid nonassociated perfectly plastic body under compatibility conditions, with respect to the velocity field. One of the fundamental aspects of the nonassociated limit analysis is the coupling between the two approaches, static and kinematic. Indeed, the functional (total plastic dissipation) to minimize in the kinematic problem contains static variables of the limit state. Conversely, the static approach contains the kinematic variables of the limit state. In this paper, the effort will be focused only to the kinematic approach that will be used to find a plastic limit state.

The presence of the stress field in the plastic dissipation is the main difficulty when we are looking for a solution to the kinematic problem. In fact, in the kinematic formulation, the stress field won't have any chance to verify neither the equilibrium equations nor the yield condition. A best solution consists of using the constitutive relations where stresses will be computed from the value of displacement field. This main difficulty will be solved by using the successive approximation algorithm in two stages [22,23]. The first one consists of keeping constant the value of the stress field and an approximation of velocity field will be obtained by solving the kinematic problem. In the second stage, the velocity field is kept constant and equal to the one obtained from the last step. A new approximation of the stress field is computed in order to satisfy the constitutive law. We notice here that the plastic bipotential is not differentiable and consequently the use of constitutive law merits to be treated. A regularization method illustrated previously in Ref. [9] allows us to overcome this difficulty. To show the feasibility of the proposed method, a simplest homogeneous deforming field is used in a simple plane strain example. Finite element method is adopted to perform an important application of punching process.

To model the nonassociative plastic flow rule of the material, we consider here the law used by Van Langen and Vermer [24], Rudnicki and Rice [25], and many other authors. This constitutive

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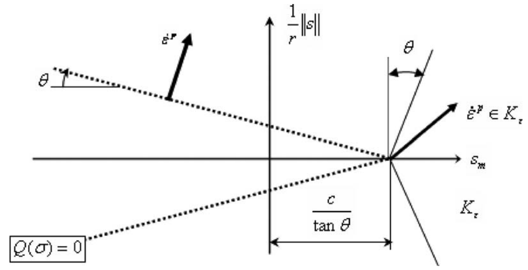

Fig. 1 Nonassociated flow rule with Drucker–Prager cone

relation uses the Drucker–Prager yield locus, which depends on two parameters, the cohesion c and the friction angle φ. The nonassociativity of the flow rule is characterized by a third parameter, the plastic dilatancy angle θ, lying within the range from 0 to φ.

2 Constitutive Relations for Nonassociated Plasticity

Let us consider that σ denoted the stress tensor within a body made of a rigid perfectly plastic material. In an usual manner, the plastic yield criterion is represented by a convex set K_σ such as

$$K_\sigma = \{\sigma \quad \text{such that} \quad f(\sigma) \leq 0\} \tag{1}$$

where $f(\sigma)$ is a mathematical function characterizing the strength of the material. In the following, a stress field σ is said to be plastically admissible (PA) if σ belongs to the set K_σ. In this paper, the Drucker–Prager model is chosen for representing the plastic strength of the material. In this model, the loading stress function $f(\sigma)$ is written in terms of the hydrostatic pressure s_m $\left(s_m = \frac{1}{3}Tr(\sigma)\right)$; the symbol Tr means trace) and the deviator stress tensor $s = \sigma - s_m I$:

$$f(\sigma) = \|s\| - r(c - s_m \tan\varphi) = 0 \tag{2}$$

where $\|s\|$ is the Euclidean norm of the deviatoric stress tensor given by $\|s\| = \sqrt{Tr(s^2)}$. The material parameters c and φ are, respectively, the cohesion and the internal angle of friction. The constant r, appearing in the Eq. (2) is related to the internal angle of friction according to the adopted matching of the Drucker–Prager criterion with the Mohr–Coulomb cone [9].

For more realistic materials, the flow rule is generally nonassociated and characterized by an additional constant material called plastic dilatancy angle, denoted as θ, within the range from 0 to φ. For any regular stress point of the plastic yielding surface, the plastic strain rate vector has an orientation defined by θ [18–20]; at the vertex $(s_m = c/\tan\varphi, \|s\| = 0)$, it belongs to the cone K_ε of vectors having an orientation less than or equal to θ (Fig. 1):

$$K_\varepsilon = \{\dot{\varepsilon}^p = (\dot{e}_m^p, \dot{e}^p) \quad \text{such that} \quad \dot{e}_m^p \geq r\tan\theta\|\dot{e}^p\|\} \tag{3}$$

where $\dot{e}_m^p$ and $\|\dot{e}^p\|$ are, respectively, the trace of strain tensor rates $\dot{\varepsilon}^p$ and the Euclidean norm of the deviator strain rate one, $\dot{e}^p$:

$$\dot{e}_m^p = Tr(\dot{\varepsilon}^p) \quad \text{and} \quad \|\dot{e}^p\| = \sqrt{Tr(\dot{e}^p)^2} \tag{4}$$

2.1 Plastic Potential. A widely used modeling of nonassociative flow rules is based on a couple of stress functions, the yield function $f(\sigma)$ to define the yield locus and Melan's plastic potential $Q(\sigma)$ to precise the direction of the plastic strain rate vector [12]:

- If $f(\sigma)=0$ and σ is a regular point of the plastic yielding surface then

$$\exists\dot{\lambda} \geq 0 \quad \text{such that} \quad \dot{\varepsilon}^p = \dot{\lambda}\frac{\partial Q(\sigma)}{\partial\sigma} \tag{5}$$

- If $f(\sigma) < 0$ then

$$\dot{\varepsilon}^p = 0 \tag{6}$$

For Drucker–Prager model, the plastic potential $Q(\sigma)$ have the following expression that differs from the yield one by the angle θ replacing φ [26]:

$$Q(\sigma) = Q(s_m, s) = \frac{1}{r}\|s\| + s_m\tan\theta - c \tag{7}$$

The application of the normality rule with Melan's plastic potential (7) leads to the nonassociated constitutive law introduced for granular materials in Ref. [21] as follows:

$$(\dot{e}_m^p + r(\tan\varphi - \tan\theta)\|\dot{e}^p\|, \dot{e}^p) \in \partial\psi_{K_\sigma}(\sigma) \tag{8}$$

where ψ_{K_σ} is the indicator function of the set K_σ defined such that $\psi_{K_\sigma}(\sigma)=0$, if $\sigma \in K_\sigma$ and $\psi_{K_\sigma}(\sigma)=+\infty$, otherwise.

Unfortunately, this approach, based on the Melan's plastic potential, does not allow extending some theories to the nonassociated flow rules for granular materials or to the Coulomb's dry friction law. Indeed, the good properties of standard materials derived from the key concepts of convexity and normality are loosed. Because these later ones are very convenient tools, the theory of the implicit standard materials (ISM) was introduced in order to extend in a natural way the good properties of the usual standard materials to the nonassociated laws as granular material plasticity and contact mechanics.

2.2 Bipotential Concept. It is based on the consideration of the normality rule, which is written in the weak form of an implicit relation (in the sense of the implicit function theorem). A scalar function b_p is said to be a bipotential if the following inequality is satisfied as follows:

$$\forall\sigma' \in K_\sigma, \quad \forall \quad \dot{\varepsilon}^{p'} \in K_\varepsilon \quad b_p(\dot{\varepsilon}^{p'}, \sigma') \geq \sigma' \,.\, \dot{\varepsilon}^{p'} \tag{9}$$

The given function b_p is separately convex with respect to the variables σ and $\dot{\varepsilon}^p$ and a pair $(\sigma, \dot{\varepsilon}^p)$ is said to be extremal if the following equality is achieved in the previous definition (9):

$$b_p(\dot{\varepsilon}^p, \sigma) = \sigma \,.\, \dot{\varepsilon}^p \tag{10}$$

meaning that the internal power for any couple $(\sigma, \dot{\varepsilon})$ is defined by the duality product where the stress σ and the plastic strain rate $\dot{\varepsilon}^p$ are related by the material law. Based on the bipotential definition (9), any extremal couple $(\sigma, \dot{\varepsilon}^p)$ is characterized by the following subdifferential relations:

$$\dot{\varepsilon}^p \in \partial_\sigma b_p(\dot{\varepsilon}^p, \sigma), \quad \sigma \in \partial_{\dot{\varepsilon}^p} b_p(\dot{\varepsilon}^p, \sigma) \tag{11}$$

In Refs. [18,21], for the Drucker–Prager model with nonassociated flow rule, one proves that the following expression is a bipotential:

If $\dot{e}_m^p \geq r\tan\theta\|\dot{e}^p\|$ and $\sigma \in K_\sigma$,

$$b_p(\dot{\varepsilon}^p, \sigma) = \frac{c}{\tan\varphi}\dot{e}_m^p + r(\tan\varphi - \tan\theta)\left(\frac{c}{\tan\varphi} - s_m\right)\|\dot{e}^p\|$$

else

$$b_p(\dot{\varepsilon}^p, \sigma) = +\infty \tag{12}$$

or, in a compact form, the last function may be rewritten in terms of indicator functions by the following expression:

$$b_p(\dot{\varepsilon}^p, \sigma) = \frac{c}{\tan\varphi}\dot{e}_m^p + r(\tan\varphi - \tan\theta)\left(\frac{c}{\tan\varphi} - s_m\right)\|\dot{e}^p\| + \psi_{K_\sigma}(\sigma) + \psi_{K_\varepsilon}(\dot{\varepsilon}^p) \tag{13}$$

where ψ_{K_ε} represents the indicator function of the set K_ε. The plastic flow rule and its inverse are expressed as an implicit normality law given by the differential inclusions (11). For more details, the reader is referred to the paper [21]. According to the expression of the bipotentials (12) or (13), one notices the presence, for the nonassociated flow rule, of a coupling term between stress and strain rate represented by the second term. This cou-

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pling term cannot be disappeared from the bipotential expression, except the event $\theta=\varphi$.

Remark. It can be noted that the last term in Eqs. (12) or (13) is the critical one in the bipotential expression because it is responsible for the coupling between stress and strain rates. The case $\theta=\varphi$ corresponds to the particular case of the associated flow rule and the bipotential is separable as follows:

If $\dot{e}_m^p \geq r \tan \varphi \|\dot{e}^p\|$ and $\sigma \in K_\sigma$,

$$b_p(\dot{\varepsilon}^p,\sigma) = V(\dot{\varepsilon}^p) + W(\sigma)$$

else

$$b_p(\dot{\varepsilon}^p,\sigma) = +\infty \tag{14}$$

with

$$V(\dot{\varepsilon}^p) = \frac{c}{\tan \varphi}\dot{e}_m^p \quad \text{and} \quad W(\sigma) = 0 \tag{15}$$

The present approach of the bipotential provides a frame theory of nonassociative constitutive laws of a kind of materials, like granular materials, radically different from the usual one using the concept of Melan's plastic potential. The law doesn't stem from two different functions, the plastic potential and the yield function, but only from a single function, the bipotential containing at once stress and strain rate. The contribution of this point of view to the limit analysis theory consists of the extension of static and kinematic approaches for nonassociated behaviors as these used for plastic flow of granular material or unilateral contact of dry friction [21]. Not only to limit analysis but also to nonassociated shakedown theory by Bousshine et al. [27], Chaaba et al. [28], and by De Saxcé et al. [29].

3 Nonassociated Limit Analysis by Kinematic Approach

On the basis of the bipotential theory, this section aims to present the founding of an extended formulation of the kinematic limit analysis approach to nonassociated plastic flow rules, in particular those obeying to the nonassociated Drucker–Prager one. First, let us clarify the limit load concept in the framework of ISM based on the bipotential approach. Recall that in the case of associated plasticity (standard plasticity) the concept of limit load is defined by the fact that such load could be achieved by an elastoplastic evolution step by step of the deformable body at stake. The limit load notion in the situation of nonassociated plasticity within the framework of the ISM may also be reached by following the elastoplastic evolution until the appearance of the plastic flow mechanism. In order to support this statement, let us consider an elastic perfectly plastic deformable body according to ISM rule subjected to body forces $\bar{f}$, to surface tractions $\bar{t}$ on a part of the boundary of the body and to kinematic boundary on the other part. The stresses tensor σ and displacement velocity vector $\dot{u}$ as a response of the continuum have to fulfill at any time, respectively, the usual equilibrium conditions (equilibrium equations and static boundary condition) and the usual compatibility conditions (strain-displacement relationships and kinematic boundary conditions), when the couple $(\sigma,\dot{\varepsilon})$ has to fulfill the elastoplastic constitutive law of the material. The elastoplastic evolution problem in the framework of ISM is based on the set of equations [19,30] such as the classical hypothesis of strain decomposition in elastoplasticity: $\dot{\varepsilon}=\dot{\varepsilon}^e+\dot{\varepsilon}^p$ and Hooke's elastic law: $\dot{\sigma}=S^{-1}\dot{\varepsilon}^e$, where S is the flexibility matrix and the plastic flow rule, which is defined as an implicit normality law through Eqs. (11) and (12), where the couple $(\sigma,\dot{\varepsilon}^p)$ is extremal for the plastic bipotential b_p. So, the history of the couple (σ,ε^p) associated with the strain history $\varepsilon(t)$ (the parameter t denotes the time in the sense of plasticity) is the solution $(\sigma(t),\varepsilon^p(t))$ of the multivalued differential equation system of first order such as $S\dot{\sigma}+\dot{\varepsilon}^p=\dot{\varepsilon}(t)$, where $(\sigma,\dot{\varepsilon}^p)$ is extremal for b_p. In order to determine the entire elastoplastic behavior, on

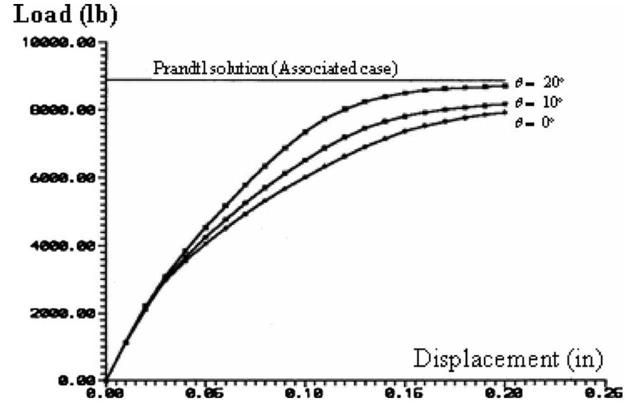

Fig. 2 Elastoplastic evolution (associated and nonassociated cases)

the basis of extended variational principles, a time integration scheme was adopted (see e.g., [19,30]). As an example of results of this elastoplastic evolution, Fig. 2 shows the elastoplastic behavior of an infinite half plane indentation, for associated and nonassociated cases, by a rigid punch with smooth border [19] using controlled displacement. In this application, the deformable body is made with a frictional elastoplastic material with Young's modulus $E=30{,}000$ psi, Poisson's coefficient $\upsilon=0.3$, a cohesion $c=10$ psi, and a friction angle $\varphi=20$ deg; the dilatancy angle θ was considered between 0 deg and 20 deg. The problem is dealt with in plane strain conditions using a quadratic finite element model. Note that at any stage of the incremental behavior, the elastoplastic law is normally fulfilled, i.e., the couple $(\sigma,\dot{\varepsilon})$ fulfills the elastoplastic behavior of the constitutive material and then by the strain-displacement relationships the couple $(\sigma,\dot{u})$ will be also. Furthermore, by following this incremental behavior step by step, by increasing the loading intensity, the plastic regions expand, merge, and the other regions (rigid or elastic) of the structure at stake could not prevent the formation of a plastic mechanism. This situation of collapse is an asymptotic state of the elastoplastic evolution called a plastic collapse or plastic flow mechanism characterized by displacement velocities $\dot{u}$ and strain rates $\dot{\varepsilon}$ that may be large under constant loading $\bar{f}^l$ and $\bar{t}^l$. The loading corresponding to this state is called the limit load. The stresses tensor σ has to be statically and plastically admissible and the velocity displacement $\dot{u}$ has to be kinematically and plastically admissible [19,30]. Consequently, this couple that characterizes the limit state verifies the principle of virtual power.

In the same line of reasoning, we can add a significant example of traction-compression in plane strain conditions [26], we dealt with the elastoplastic behavior by means of a semi-analytic calculation and a finite element computation in the situation of nonassociated flow rule with the Drucker–Prager criterion. It was shown that the limit load obtained in a direct way using the approach presented in this paper coincides exactly with the limit load obtained by the elastoplastic computation; this is valid for all cases tested of nonassociated situations. Furthermore, in the framework of shakedown analysis, the asymptotic solution from an elastoplastic evolution and the direct bearing capacity load do not seem to depend on the initial state (see Remark on page 17 of Ref. [31]). If this is true for the shakedown analysis, we think that it will be also for the limit analysis as a particular case of the previous one. Hence, it may be concluded from these applications of elastoplastic testing that the plastic limit state and then collapse loads would not depend on initial conditions and the elasticity when the flow rule is nonassociated as it is true for the associated case. We can also add that within a plastic limit state for rigid perfectly plastic materials, related to a plastic flow mechanism, limit load, and stresses become constant despite of displacement

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velocity and strain rates evolution. Because the linear feature of the relationship between stresses and elastic strains part via Hooke's law ($\dot{\varepsilon}=S\dot{\sigma}+\dot{\varepsilon}^p$), we may state that the elastic strain rates will vanish if stresses within the body are constant, this is the case with the plastic limit state. This observation remains valid and independent of the plastic behavior of the material if it is associated or nonassociated. In the sequel, the plastic part of strain rates tensor is omitted.

To close this discussion on the validity of the limit load concept in the situation of the nonassociated case, referring to a previous paper (see Ref. [22]), which dealt with nonassociated limit analysis of plane frames with frictional contact support by adopting the same approach of the bipotential concept as presented in this paper, limit loads obtained by the kinematic and static formulations are exactly identical. These results show that the limit loads are correct and the kinematic and static formulations are dual in the sense that the lower bound is exactly equal to the upper one.

Come back to the limit analysis framework where the common starting point of kinematic limit analysis approach consists of the consideration of admissible velocity fields. Let us consider now Ω as a rigid perfectly plastic body of boundary S subjected to body forces $\bar{f}$ and to surface tractions $\bar{t}$ on the part S_1 of S. On the other hand, it is subjected to zero velocity on the second part S_0 of S. Besides, as usual in limit analysis, the body forces and surface tractions are assumed to be proportional to the reference loads such as $\bar{f}=\alpha\bar{f}^0$ and $\bar{t}=\alpha\bar{t}^0$, where $\bar{f}^0$ and $\bar{t}^0$ are fixed and represent the reference loading with a positive number α called load factor. To begin with, let us recall some definitions.

A velocity field $\dot{u}^k$ is said to be kinematically admissible (KA) if the following compatibility conditions are fulfilled.

$$\dot{\varepsilon}(\dot{u}^k)=\text{grad}_s\,\dot{u}^k \quad \text{within} \quad \Omega \quad \text{and} \quad \dot{u}^k=0 \quad \text{on} \quad S_0 \tag{16}$$

On the other hand, a stress field σ^s is said to be statically admissible (SA) if the following equilibrium equations are satisfied.

$$\text{div}\,\sigma^s=-\alpha^s\bar{f}^0 \text{ within } \Omega \text{ and } t(\sigma^s)=\sigma^s\cdot n=\bar{t}=\alpha^s\bar{t}^0 \text{ on } S_1 \tag{17}$$

In Eq. (17), α^s is a non-negative real number called static factor and associated to the SA stress tensor through Eq. (17). For any SA stress field σ^s and KA velocity field $\dot{u}^k$, the usual principle of virtual power can be stated in the following form:

$$\int_\Omega \sigma^s\,.\,\dot{\varepsilon}(\dot{u}^k)d\Omega=\int_\Omega \dot{u}^k\,.\,\bar{f}d\Omega+\int_{S_1}\dot{u}^k\,.\,\bar{t}dS \tag{18}$$

3.1 Limit State and Load Factors. As it was discussed in the beginning of Sec. 3, let $\bar{t}^l$ and $\bar{f}^l$ be, respectively, the loading of surface tractions and body forces causing the plastic collapse of a given structure. Recall that this limit loading could be reached by an elastoplastic way step by step as it was shown previously. For the limit state, the collapse mechanism is characterized by a velocity field $\dot{u}$ and a static one σ within the structure at stake. The elementary internal power, represented by the plastic dissipation, for the limit state, is equal to the product $\sigma.\dot{\varepsilon}$. On the other hand, the elementary external power created by the external loading $\bar{t}^l$ and $\bar{f}^l$ for the velocity field $\dot{u}$ is composed, respectively, of the two terms $\dot{u}.\bar{t}^l$ and $\dot{u}.\bar{f}^l$.

Consequently, by taking into account that the couple $(\sigma,\dot{\varepsilon})$ is extremal for the bipotential defined by the relation (10), the principle of virtual power can be stated as follows:

$$\int_\Omega b_p(\dot{\varepsilon}(\dot{u}),\sigma)d\Omega=\int_{S_1}\dot{u}\,.\,\bar{t}^l dS+\int_\Omega \dot{u}\,.\,\bar{f}^l d\Omega \tag{19}$$

We insist on the fact that the so-called limit state must not be trivial in the sense that an external positive power is dissipated plastically within the structure. Considering a non-negative load factor, the formulation (18) leads, for any kinematically admissible velocity field $\dot{u}^k$ to

$$\int_{S_1}\dot{u}^k\,.\,\bar{t}^0 dS+\int_\Omega \dot{u}^k\,.\,\bar{f}^0 d\Omega>0 \tag{20}$$

Let α^l be the load factor such that $\bar{t}^l=\alpha^l\bar{t}^0$ and $\bar{f}^l=\alpha^l\bar{f}^0$. The positive number α^l is called limit load factor and it can be defined without ambiguity by the previous equality such that

$$\int_\Omega b_p(\dot{\varepsilon}(\dot{u}),\sigma)d\Omega=\alpha^l\left[\int_{S_1}\dot{u}\,.\,\bar{t}^0 dS+\int_\Omega \dot{u}\,.\,\bar{f}^0 d\Omega\right] \tag{21}$$

or, because of $\dot{u}$ is an admissible velocity field:

$$\alpha^l=\frac{\int_\Omega b_p(\dot{\varepsilon}(\dot{u}),\sigma)d\Omega}{\int_{S_1}\dot{u}\,.\,\bar{t}^0 dS+\int_\Omega \dot{u}\,.\,\bar{f}^0 d\Omega} \tag{22}$$

By analogy with the definition of the limit load factor, the kinematic one, denoted α^k, associated to a velocity field $\dot{u}^k$, characterizing an admissible mechanism is defined by the following relation:

$$\alpha^k=\frac{\int_\Omega b_p(\dot{\varepsilon}(\dot{u}^k),\sigma)d\Omega}{\int_{S_1}\dot{u}^k\,.\,\bar{t}^0 dS+\int_\Omega \dot{u}^k\,.\,\bar{f}^0 d\Omega} \tag{23}$$

Because the bipotential is a homogeneous function of order one with respect to the velocity field $\dot{u}^k$, the normalization condition, usually adopted in this kind of problems, is used. So, it allowed imposing a normalization condition as usual:

$$\int_{S_1}\dot{u}^k\,.\,\bar{t}^0 dS+\int_\Omega \dot{u}^k\,.\,\bar{f}^0 d\Omega=1 \tag{24}$$

Then, the kinematic load factor is identical to the virtual dissipation:

$$\alpha^k=\int_\Omega b_p(\dot{\varepsilon}(\dot{u}^k),\sigma)d\Omega \tag{25}$$

3.2 Upper Bound Problem for the Nonassociated Plastic Flow. The limit analysis theory is one of direct fundamental methods allowed to find the plastic limit state without performing complete elastoplastic computations. The background of this analysis constitutes limit analysis theorems, static and kinematic ones, on which the method can be applied effectively to engineering problems. On the basis of the bipotential theory, the two limit analysis theorems are extended to the nonassociated case. In the present work, only the kinematic approach will be treated. Using constitutive relations and the definition of the bipotential concept, the kinematic approach is based on the following kinematic proposition [21]: Let α^l be a limit load factor associated to a limit state $(\dot{u},\sigma)$ and α^k a kinematic load factor associated to an admissible velocity field, $\dot{u}^k$, one has $\alpha^k\geq\alpha^l$.

Based on the previous kinematic theorem and the definition (23) of the kinematic load factor and taking into account the normalization condition (24), the limit state characterized by a couple $(\dot{u},\sigma)$ can be determined by solving the following nonlinear mathematical programming problem:

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$$\text{Inf} \quad \int_\Omega b_p(\dot\varepsilon(\dot u^k),\sigma)d\Omega$$

$$\text{subject to:} \quad \dot u^k \quad \text{admissible}$$

$$\int_{S_1} \dot u^k \,.\, \bar t^0 dS + \int_\Omega \dot u^k \,.\, \bar f^0 d\Omega = 1 \tag{26}$$

In this mathematical program, the stress field σ is kept constant to an optimal value corresponding to the limit state when the velocity field $\dot u^k$ represents the unknowns of the problem. For the nonassociated Drucker–Prager model, the kinematical bound problem takes the explicit form:

$$\text{Inf} \quad \int_\Omega \left[\frac{c}{\tan\varphi}\dot e_m(\dot u^k) + r(\tan\varphi - \tan\theta)\left(\frac{c}{\tan\varphi} - s_m\right)\|\dot e(\dot u^k)\|\right]d\Omega$$

$$\text{subject to:} \quad \dot u^k \quad \text{admissible}, \quad \dot e_m \geq r\tan\theta\|\dot e\| \quad \text{in} \quad \Omega$$

$$\int_{S_1} \dot u^k \,.\, \bar t^0 dS + \int_\Omega \dot u^k \,.\, \bar f^0 d\Omega = 1 \tag{27}$$

In the above constrained optimization problem, the constraint "$\dot u^k$ must be admissible" means that two conditions need to be satisfied. The first one is that strain rates must be derived from the continuous velocity field $\dot u^k$, which is supposed to verify boundary conditions. The second constraint imposes the implicit normality law. For realistic problems, the above mathematical program will be solved by numerical procedures as in general finite elements method. Some aspects require taking care of; the first one is the presence of statical variables in the kinematic approach. This statement is appropriate to the nonassociated flow rules. It is clear that the problem becomes more complicated in comparison with the standard formulation derived from the associated flow rule. The second aspect is the no smooth nature of the objective function formulated by the bipotential. It may be remarked that when $\theta=\varphi$ (standard material), the flow rule is associated and the coupling term appearing in the objective function in the above mathematic programming problem will disappear.

$$\text{Inf} \quad \int_\Omega \frac{c}{\tan\varphi}\dot e_m(\dot u^k)d\Omega$$

$$\text{subject to:} \quad \dot u^k \quad \text{admissible}, \quad \dot e_m \geq r\tan\varphi\|\dot e\|$$

$$\int_{S_1} \dot u^k \,.\, \bar t^0 dS + \int_\Omega \dot u^k \,.\, \bar f^0 d\Omega = 1 \tag{28}$$

4 Coupling Problem

For the nonassociated plastic behavior, the coupling term between stresses and strain rates is usually present in the plastic bipotential. Naturally, the use of the kinematic approach leads automatically to the use of kinematically admissible approximations as finite element method. So, the stress field appearing in the mathematical programming problem will not be discretized and it has no chance to verify neither equilibrium equation nor the plasticity criterion. But, the constitutive relations relating stresses to strain rates may be used. Recalling that theses relations are expressed as a subdifferential implicit law, $\sigma \in \partial_{\dot\varepsilon} b_p(\dot\varepsilon,\sigma)$, which needs to be satisfied anywhere in the body. Then, the stress field can be computed approximately when the velocity field is known. Finally, the coupling problem will be solved iteratively by a successive approximation algorithm as explained by the following way.

We assume that we start from an initial state characterized by the couple $(\dot u,\sigma)$ with null value. Let $(\dot u,{}^i\sigma^i)$ denotes the value of stresses and velocities at the ith iteration. It is assumed to be known at the current iteration. The value of the couple $(\dot u,\sigma)$ at the next iteration will be determined in two times, as follows:

- In this first step, the stress field appearing in the objective function to minimize is taken as a simple parameter which kept constant during this step having σ^i as a value. In this time, the value of strain rate $\dot u^{i+1}$ is obtained by solving the kinematical problems (26) or (27) for the Drucker–Prager model. A package software of optimization computation is used for solving the nonlinear mathematical programming problem having the velocity field as variables.
- In the second time, this is the stress field that will be computed by means of the constitutive relations (11) by keeping the velocity field constant and equal to $\dot u^{i+1}$ obtained in the previous step.

This iterative process will be repeated as far as convergence using one of usual criteria. This algorithm concerned with the solving of the coupling question between stresses and strain rates or velocities is applied in the case of the determination of the limit state of frames with presence of frictional contact supports [22]. In the present paper, the same procedure will be applied to a simple loading body in plane strain conditions (see Sec. 6).

5 Regularization of the Plastic Bipotential

The plastic bipotential, like the plastic dissipation, constituting the objective function of the kinematic problems (26) or (27) is only subdifferentiable. Further, the stress filed is given by means of a differential inclusion. To use correctly package software of optimization computation that uses deterministic algorithms, the nonlinear objective function must be differentiable. In fact, package software requires to precise the gradients of the objective function. For this reason, we need to regularize the objective function. In this section, we present a procedure to regularize the plastic bipotential by following the same steps used for standard materials presented in a previous paper [9]. The regularized model used here consists to replace the rigid perfectly plastic behavior by a fictitious linear viscous perfectly plastic one considering that the total strain rate $\dot\varepsilon$ is divided in two parts:

$$\dot\varepsilon^v + \dot\varepsilon^p = \dot\varepsilon \tag{29}$$

where $\dot\varepsilon^v$ is a fictitious viscous strain rate related to stresses by $\dot\varepsilon^v = S_v\sigma$, with the matrix S_v is analogous to the elastic stiffened one whereas the plastic part $\dot\varepsilon^p$ is given by the normality law (11). The regularization method used here is based on the convolution operation, which uses an auxiliary function that is deduced from the linear viscous strain rate defined by the following separated bipotential:

$$b_v(\dot\varepsilon^v,\sigma) = V(\dot\varepsilon^v) + W(\sigma) \tag{30}$$

with

$$\begin{aligned} V(\dot\varepsilon^v) &= \frac{1}{2}\dot\varepsilon^v \,.\, \sigma = \frac{1}{2}\dot\varepsilon^v S_v^{-1}\dot\varepsilon^v = \frac{K_c^*}{2}(\dot e_m^v)^2 + \mu^*\|\dot e^v\|^2 \\ W(\sigma) &= \frac{1}{2}\sigma \,.\, \dot\varepsilon^v = \frac{1}{2}\sigma S_v\sigma = \frac{1}{2K_c^*}(s_m)^2 + \frac{1}{4\mu^*}\|s\|^2 \end{aligned} \tag{31}$$

where the decomposition $\dot\varepsilon^v = (\dot e_m^v, \dot e^v)$ is considered. The symbol $\|.\|$ being the norm defined by $\|\dot\varepsilon\|^2 = \frac{1}{2}(\dot\varepsilon S_v \dot\varepsilon)^{\frac{1}{2}}$. The parameters K_c^* and μ^* are fictitious constants and must be estimated for each problem. It is easy to show that the function defined by Eq. (30) associated to the linear viscous law is a bipotential. The linear viscous law and its inverse can be expressed as a normality rule such that

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$$\sigma \in \partial_{\dot{\varepsilon}^v} b_v(\dot{\varepsilon}^v, \sigma) \quad \text{and} \quad \dot{\varepsilon}^v \in \partial_\sigma b_v(\dot{\varepsilon}^v, \sigma) \tag{32}$$

or because the bipotential b_v is a differentiable function the previous differential inclusion becomes

$$\sigma = \frac{\partial b_v}{\partial \dot{\varepsilon}^v} = \frac{\partial V}{\partial \dot{\varepsilon}^v}, \quad \text{and} \quad \dot{\varepsilon}^v = \frac{\partial b_v}{\partial \sigma} = \frac{\partial W}{\partial \sigma} \tag{33}$$

The couple $(\dot{\varepsilon}^v, \sigma)$ satisfying the last relation is qualified as extremal for the bipotential b_v and we have the following equality:

$$\dot{\varepsilon}^v \, . \, \sigma = b_v(\dot{\varepsilon}^v, \sigma) = V(\dot{\varepsilon}^v) + W(\sigma) \tag{34}$$

to simplify the presentation and because $W(\sigma)$ is only function of stresses, this part of $b_v(\dot{\varepsilon}^v, \sigma)$ is omitted in the following. Let b be the real function of two variables, the strain rate $\dot{\varepsilon}$ and the stress field σ, obtained by the *inf*-convolution [9] of b_v and b_p with respect to the plastic strain rate $\dot{\varepsilon}^p$. Taking into account that $\dot{\varepsilon}^v = \dot{\varepsilon} - \dot{\varepsilon}^p$, the regularized bipotential is obtained by

$$b(\dot{\varepsilon}, \sigma) = \operatorname*{Inf}_{\dot{\varepsilon}^p} \{ b_v(\dot{\varepsilon} - \dot{\varepsilon}^p, \sigma) + b_p(\dot{\varepsilon}^p, \sigma) \}$$
$$\text{subject to:} \quad \dot{e}_m^p \geq r \tan \theta \| \dot{e}^p \| \quad \text{in} \quad \Omega \tag{35}$$

with the stress field σ belongs to the set of plastically admissible stresses. For the associated or the nonassociated Drucker–Prager or von Mises materials, the minimum of Eq. (35) with respect to the plastic strain rate $\dot{\varepsilon}^p$ can be easily determined analytically. By excluding the vertex of the Drucker–Prager cone and supposing that the equality is reached in the mathematical constraint of Eq. (35), $\dot{e}_m^p$ is eliminated from Eq. (35) and the regularized bipotential takes the explicit form:

$$b(\dot{\varepsilon}, \sigma) = \operatorname*{Inf}_{\dot{e}^p} \left\{ \frac{K_c^*}{2} (\dot{e}_m - r \tan \theta \| \dot{e}^p \|)^2 + \mu^* \| \dot{e} - \dot{e}^p \|^2 + r[c - (\tan \varphi - \tan \theta) s_m] \| \dot{e}^p \| \right\} \tag{36}$$

The stationary condition in Eq. (36) corresponding to the optimum with respect to the plastic strain deviator $\dot{e}^p$ provides the following relation:

$$2\mu^* \dot{e} = \left\{ 2\mu^* + \frac{1}{\| \dot{e}^p \|} [K_c^* r \tan \theta (r \tan \theta \| \dot{e}^p \| - \dot{e}_m) + r(c - (\tan \varphi - \tan \theta) s_m)] \right\} \dot{e}_p \tag{37}$$

which shows that the vectors $\dot{e}^p$ and $(2\mu^* \dot{e})$ are collinear, and $\dot{e}^p = \| \dot{e}^p \| \hat{n}$, where $\hat{n}$ is the unitary vector in the direction of deviatoric rate tensor $\dot{e}$ defined by $\dot{e}/\|\dot{e}\|$. By making equal the norms of the two hands of Eq. (37), one obtains easily:

$$\| \dot{e} \| \geq \varepsilon_d [c - (\tan \varphi - \tan \theta) s_m - K_c^* \tan \theta \dot{e}_m)]$$

$$b(\dot{\varepsilon}, \sigma) = \frac{K_c^*}{2} (\dot{e}_m)^2 + \mu^* \left\{ \| \dot{e} \|^2 - \frac{1}{1 + \varepsilon_c r^2 \tan^2 \theta} (\| \dot{e} \| - \varepsilon_d [c - (\tan \varphi - \tan \theta) s_m - K_c^* \tan \theta \dot{e}_m])^2 \right\}$$

$$b(\dot{\varepsilon}, \sigma) = \frac{K_c^*}{2} (\dot{e}_m)^2 + \mu^* \| \dot{e} \|^2 \tag{38}$$

where the constants ε_c and ε_d are introduced for convenience such as $K_c^*/(2\mu^*)$ and $\varepsilon_d = r/(2\mu^*)$. Finally, the mathematical programming problems (26) or (27) representing the kinematic approach is modified as follows:

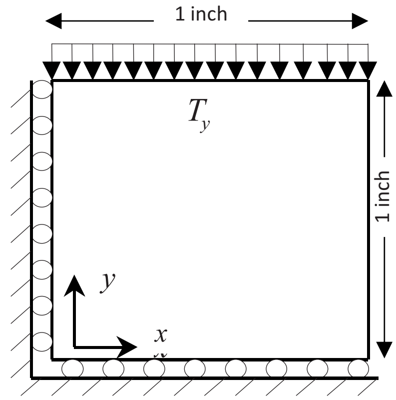

Fig. 3 Geometry and boundary conditions

$$\text{Inf} \quad \int_\Omega b(\dot{\varepsilon}(\dot{u}^k), \sigma) d\Omega$$
$$\text{subject to:} \quad \dot{u}^k \quad \text{admissible}$$
$$\int_{S_1} \dot{u}^k \, . \, \bar{t}^0 dS + \int_\Omega \dot{u}^k \, . \, \bar{f}^0 d\Omega = 1 \tag{39}$$

Remark. The particular case, corresponding to the associated Drucker–Prager model, is obtained by considering the dilatancy angle be equal to the internal angle of friction:

$$b(\dot{\varepsilon}, \sigma) = \frac{K_c^*}{2} (\dot{e}_m)^2 + \mu^* \left\{ \| \dot{e} \|^2 - \frac{1}{1 + \varepsilon_c r^2 \tan^2 \varphi} (\| \dot{e} \| - \varepsilon_d [c - K_c^* \tan \varphi \dot{e}_m])^2 \right\} \tag{40}$$

For the von Mises model, the regularized bipotential is obtained by considering the dilatancy angle and the internal angle of friction to be equal to zero:

$$b(\dot{\varepsilon}, \sigma) = \frac{K_c^*}{2} (\dot{e}_m^2) + \mu^* \{ \| \dot{e} \|^2 - (\| \dot{e} \| - c \varepsilon_d)^2 \} \tag{41}$$

A simple illustration of the presented regularized method to solve the upper bound kinematical problem is shown in a previous paper dealing with kinematic limit analysis in the case of standard material [9].

6 Simple Traction and Compression of a Square Block

Let us consider a square block (Fig. 3) of granular material with edges parallel to the x, y, and z axis subjected to traction and compression actions on its edges, under plane strain conditions, between two rigid and smooth plates. The material properties of the material used for numerical computation are $c = 30$ psi, $\varphi = 40$ deg, and $\theta = 0$ deg.

The structure is supposed to be subjected, by means of the two rigid smooth plates, to a pressure distributed uniformly on the y direction, $T_y = \alpha T_0$. Because the area surfaces are equal to the unity, the stress σ_y is constant in the body and related to a reference stress σ_y^0 such as $\sigma_y = \alpha \sigma_y^0$, where σ_y^0 may be chosen as $\sigma_y^0 = 1$ for the traction and $\sigma_y^0 = -1$ for the compression case. Because the lateral edges are free, the stress σ_x in the direction x is supposed to be zero. By considering the simplest homogeneous deforming fields, the strain rates are constant in the specimen and equal to their corresponding horizontal and vertical displacement rates: $\dot{\varepsilon}_x = \dot{u}_x$ and $\dot{\varepsilon}_y = \dot{u}_y$. For this simple application, the regularized kinematic problem can be expressed as follows:

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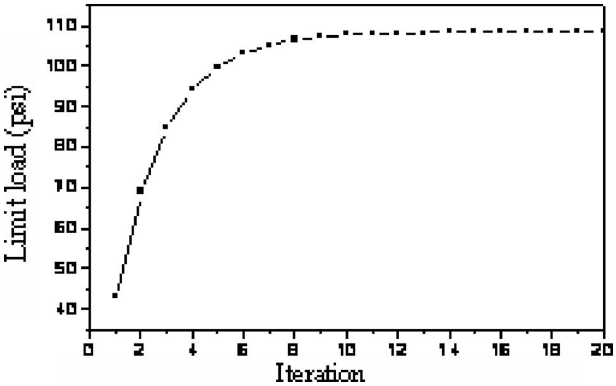

Fig. 4 Limit load: compression case

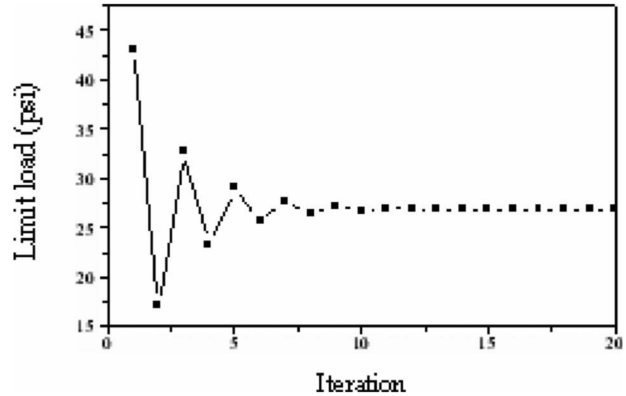

Fig. 6 Limit load: traction case

$$\text{Inf} \quad \int_{\Omega} b(\dot{\varepsilon}(\dot{u}),\sigma)d\Omega$$

$$\text{subject to:} \quad \dot{u} \quad \text{admissible} \qquad \sigma_x^0\dot{u}_x + \sigma_y^0\dot{u}_y = 1 \tag{42}$$

$$\text{with the constitutive relations:} \quad \sigma_x = \frac{\partial b}{\partial \dot{u}_x}, \quad \sigma_y = \frac{\partial b}{\partial \dot{u}_y} \tag{43}$$

where the loading references σ_x^0 and σ_y^0 are such that σ_x^0 is assumed to be equal zero and $\sigma_y^0 = \pm 1$.

The initial value of the couple $(\dot{u},\sigma)$ is taken equal zero. By applying the iterative process presented in Sec. 4, limit loads and stresses are computed. They are obtained by the approached constitutive relations (43). The evolution of the limit loads and stresses during iterations for compression case are illustrated in Figs. 4 and 5, For the traction case, these variables are presented in Figs. 6 and 7.

It may be remarked that for the traction case, the limit load and the stresses in the body at limit state converge to the analytic ones by increasing with oscillation. On the other hand, for the compression case, theses magnitudes converge without oscillating. The limit loads, by numeric approach, are plotted later with respect to the nonassociativity ratio $\rho = \tan\theta/\tan\varphi$.

7 Numerical Computation

To perform practical problems, the finite element method is adopted. As usual in kinematic approaches, the discretization of the displacement field is defined by the relation:

$$\dot{u}(x) = N(x)\dot{U} \tag{44}$$

where $\dot{U}$ is the unknown nodal velocity vector and $N(x)$ is the matrix of polynomial shape function. By considering the compatibility conditions, the strain tensor rate field is defined by

$$\dot{\varepsilon}(x) = B(x)\dot{U} \tag{45}$$

where $B(x)$ is the classic matrix composed by the first derivatives of shape functions with respect to space variables: $B(x) = \text{grad}_s\, N(x)$. In the following, we suppose that the structure is discretized in a number ne of finite elements. The regularized bipotential (38) is given by the following discretization form:

$$B(\dot{U},\sigma) = \int_{\Omega} b(\dot{\varepsilon},\sigma)d\Omega = \sum_{i=1}^{ne}\int_{\Omega_e} b(B(x)\dot{U},\sigma)d\Omega \tag{46}$$

On the other hand, the discretized form of the normalization condition in terms of nodal velocity vector is

$$F_0^T\,.\,\dot{U} = \int_{\Omega} \dot{u}\,.\,\vec{f}^0 d\Omega + \int_{S_1} \dot{u}\,.\,\vec{t}^0 dS = \sum_{i=1}^{ne}\int_{\Omega_e} \dot{u}\,.\,\vec{f}^0 d\Omega + \sum_{i=1}^{n\sigma e}\int_{S_{\sigma e}} \dot{u}\,.\,\vec{t}^0\,.\,dS = 1 \tag{47}$$

With $n\sigma e$ is the number of finite elements of the boundary, where forces are imposed and $S_{\sigma e}$ represents the length of a finite element on the boundary S_1. F_0 is a reference imposed nodal forces on the part S_1 of the boundary, it is defined by

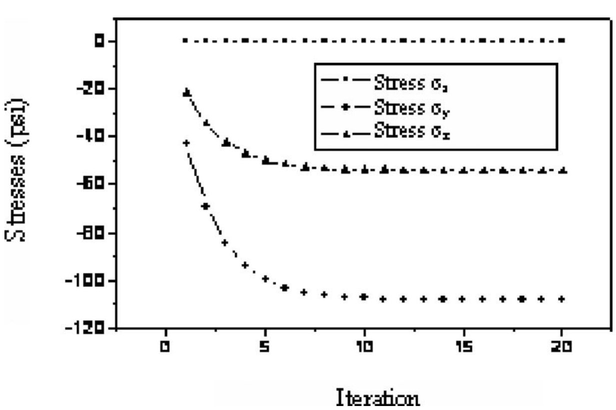

Fig. 5 Stresses: compression case

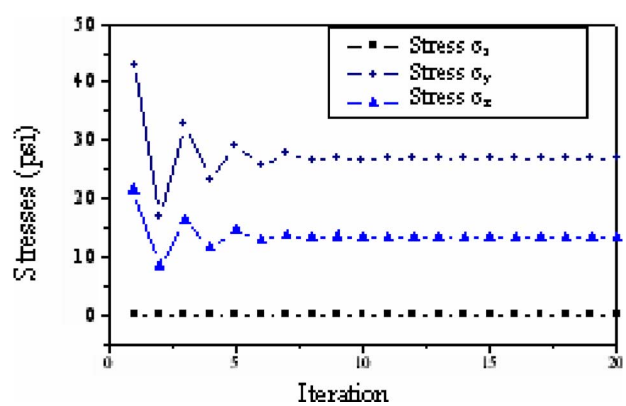

Fig. 7 Stresses: traction case

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$$F_0 = \int_\Omega N^T \overline{f}^0 d\Omega + \int_{S_1} N^T \overline{t}^0 dS \tag{48}$$

As mentioned before in Sec. 4, the couple $(\dot{U}, \sigma)$ characterizing the limit state will be computed by an iterative procedure in two steps. Starting with the fact that the couple $(\dot{U}^i, \sigma^i)$ is known and after, we are looking for $(\dot{U}^{i+1}, \sigma^{i+1})$:

1. σ^i is fixed and a new approximation $\dot{U}^{i+1}$ is computed by solving the following minimization problem by a nonlinear mathematical programming algorithm. To deal with, MINOS software [32] is used.

$$\text{Inf} \quad \int_\Omega b(B(x)\dot{U}, \sigma) d\Omega$$

$$\text{subjected to:} \quad F_0^T \dot{U} = 1 \tag{49}$$

2. $\dot{U}^{i+1}$ is fixed and a new approximation σ^{i+1} is computed by using constitutive relations. Taking into account that the regularized bipotential is differentiable:

$$\dot{U}^{i+1} = \frac{\partial b(\dot{U}^{i+1}, \sigma^i)}{\partial \varepsilon} \tag{50}$$

Remark. In order to obtain a well conditioned optimization problem by a suitable scaling of the unknown vector $\dot{U}^{i+1}$ and the objective function $B(\dot{U}, \sigma)$, we consider the following scaling:

$$\bar{\dot{U}} = \dot{U}/\dot{U}_R \quad \text{and} \quad \bar{B} = B/B_R \tag{51}$$

where $\dot{U}_R$ and B_R are estimated as follows:

$$\dot{U}_R = \frac{F_R}{2E_r^*} \quad \text{and} \quad B_R = F_R U_R \tag{52}$$

with F_R is a reference imposed force on the boundary S_1 and E_r^* is a fictitious reference Young's modulus.

7.1 Traction and Compression of a Square Block. We consider the same square block, showed in Fig. 3, for which an exact solution is presented before (see Sec. 6 and Ref. [26]). Recalling that the specimen in plane strain condition obeys to the Drucker–Prager model and nonassociated flow rule with the following properties: cohesion: $c=30$ psi, an internal angle of friction: $\varphi=40$ deg, the dilatancy angle θ may take values between 0 deg and 40 deg. The domain is discretized by isoparametric finite elements with three nodes triangles. Because the stresses are homogenous in the specimen, we use only two finite elements. Limit loads found by the presented algorithm are plotted at Figs. 8 and 9, for different values of θ and for the cases, respectively, traction and compression. Even we used only two finite elements, the result graphics show a very good agreement between analytic and numeric solutions and this is true for all values of the dilatancy angle between 0 deg and 40 deg. Fictitious values used in this computation are $E^*=10^6$ and $\nu^*=0.3$.

7.2 Infinite Half-Plane Punching. One of the most interesting problems for testing numerical results is the infinite half plane punching process satisfying the plane strain condition. For this, consider a frictional plastic material characterized by a cohesion $c=10$ psi and a friction angle $\varphi=20$ deg or $\varphi=30$ deg. The dilatancy angle θ is between 0 deg and φ. Taking into account the symmetry, the geometry, the loading, and boundary conditions used here are showed in Fig. 10 below. The loading considered here is assumed to be only a pressure uniformly distributed beneath the punch. The mesh used for this computation is showed in Fig. 11 with linear (T3) or nonlinear (T6) triangular finite elements.

As usual in this kind of problems, we define the bearing capacity factor N_c by the limit load divided by the half-width of the punch and by the cohesion. Limit loads in terms of the bearing capacity factor found by the present algorithm for different values of the angle θ between 0 deg and 30 deg are plotted in Fig. 12 using the mesh of 99 T3. In the case of using the mesh of 99 T6, N_c is plotted in Fig. 13 but only for $18 \text{ deg} \leq \theta \leq 30 \text{ deg}$. In fact, we remarked that, for used nonlinear finite elements, the algorithm encountered some convergence problems when a high accuracy of the convergence criterion is specified on the constitutive laws. This is true for dilatancy angle values less than about 18

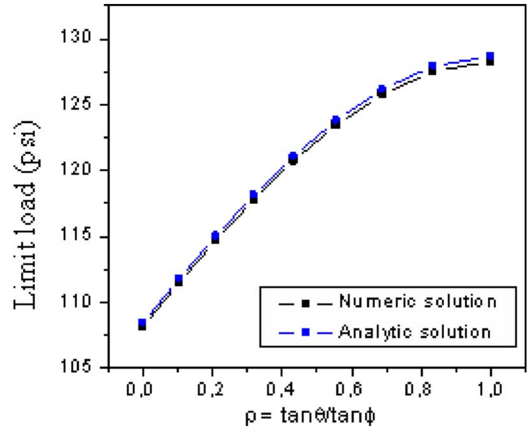

Fig. 8 Limit loads (comparison of result): compression

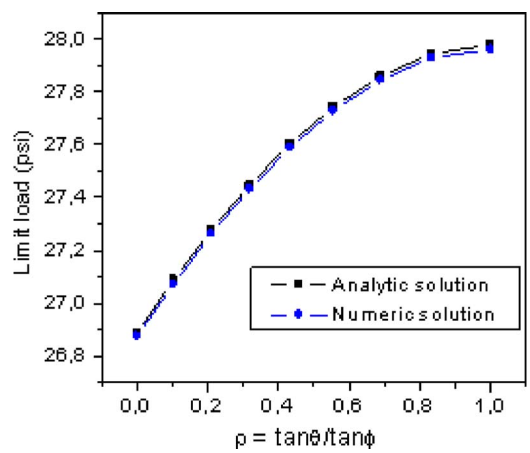

Fig. 9 Limit loads (comparison of result): traction

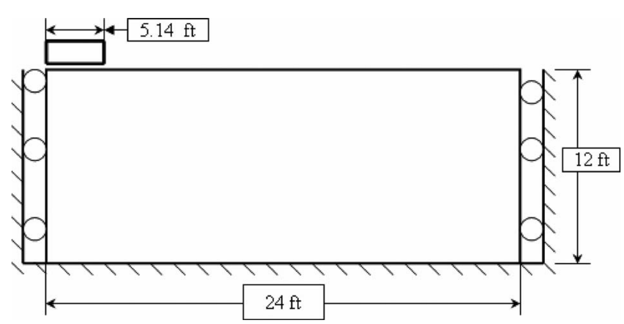

Fig. 10 Mathematical model: geometry and boundary conditions

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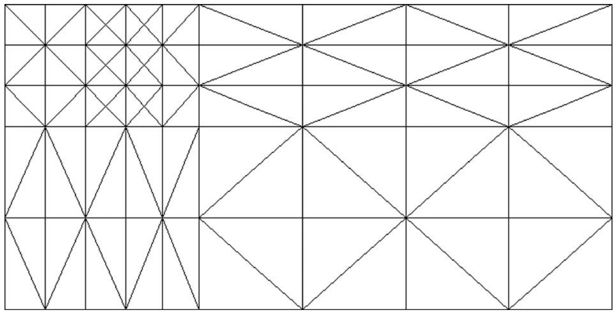

Fig. 11 Mesh used: 99 linear triangular finite elements

deg. The convergence aspect using the mesh of 99 T3, for an example of nonassociative case ($\varphi=30$ deg and $\theta=10$ deg), is showed in Fig. 14.

From all these results it can be noted that the limit load of a nonassociative material ($\theta<\varphi$) is always less than the one corresponding the standard case ($\theta=\varphi$). The variation way of the limit loads with respect to the dilatancy angle is the same for the all examples as presented previously (square block and punching process). The comparison between the degree of the finite element show that the linear kind is more adequate from point of view of convergence but the nonlinear kind is more economic. Then, the use of linear finite element mesh requires in general refining meshes.

Keeping all things the same, only we change the internal angle of friction to 20 deg, Figures 15 and 16 present the bearing capacity factor with respect to the dilatancy angle using respectively 99 linear triangular element T3 and 99 T6. The obtained results confirm what we already said for the case of $\varphi=30$ deg.

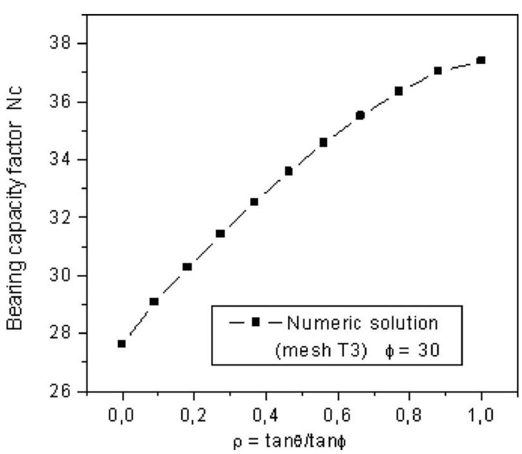

Fig. 12 Bearing capacity factor Nc: mesh of 99 linear T3

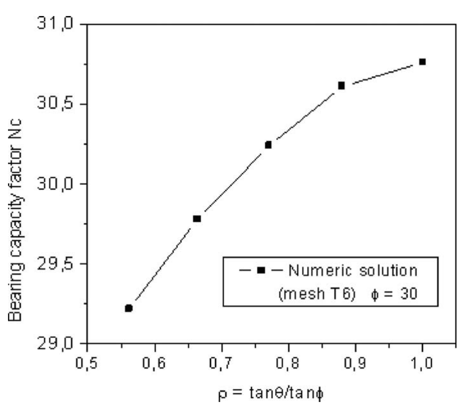

Fig. 13 Bearing capacity factor Nc: mesh of 99 nonlinear T6

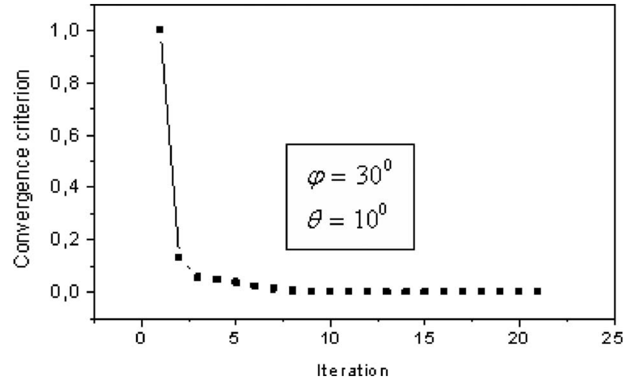

Fig. 14 Error norm of stress as function of number of iterations $\varphi=30$ deg and $\theta=10$ deg

Figures 17 and 18 show the collapse mechanisms using velocity field for the two cases ($\varphi=30$ deg and $\theta=30$ deg) and ($\varphi=30$ deg and $\theta=10$ deg).

8 Conclusion

The application of limit analysis by the use of kinematic approach to nonassociated rigid perfectly plastic materials has been

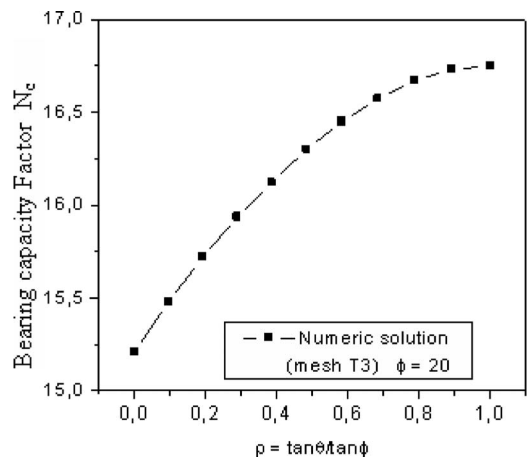

Fig. 15 Bearing capacity factor Nc: Mesh with 99 linear T3

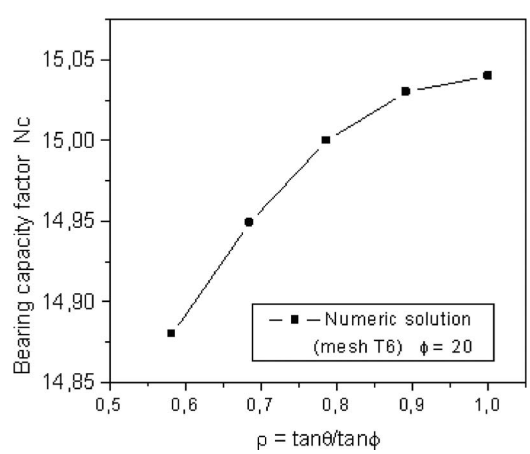

Fig. 16 Bearing capacity factor Nc: Mesh with 99 nonlinear T6

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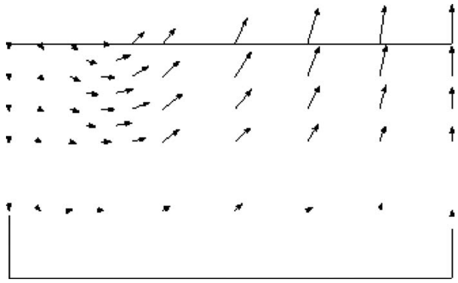

Fig. 17 Velocity field for φ=30 deg and θ=30 deg (99 T3)

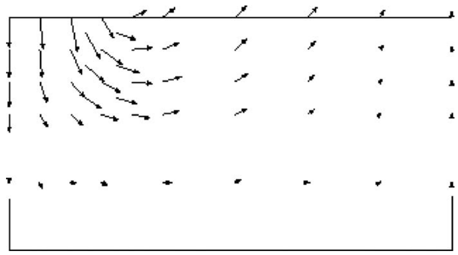

Fig. 18 Velocity field for φ=30 deg and θ=10 deg (99 T3)

investigated. The plastic bipotential representing the plastic dissipation has been not differentiable and contains a coupling term characterizing the nonassociated standard behavior. With the help of the *inf*-convolution concept, we are able to avoid difficulties caused by the nondifferentiability of the plastic dissipation. Further, this mathematical tool allows us to deduce, in an approximate manner, the stress field by using the constitutive relations, as an approached gradient of the objective function. The application of this algorithm on a simple academic problem shows the feasibility of these procedures. The use of the finite element method allows us to perform important practical problems like punching. Then, we showed that how the limit load varies with respect to the dilatancy angle. Difficulties related to the convergence of the algorithm for some values of the dilatancy angles and nonlinear finite elements need more attention. To close this conclusion, we mention that the old question of nonuniqueness connected to the nonassociated plasticity discussed, e.g., in Refs. [33,34] is not addressed in this paper. In a previous work using two pads [35], it has been shown that the limit load could be not unique. However, it is a very particular example and we do not think we can draw general conclusions. This important issue deserves further discussion and remains an open question.

Nomenclature

f = loading function stress
b = bipotential
s = stress deviator
s_m = hydrostatic pressure
u = displacement vector
Ω = structure volume
S_0 = part of the boundary S of Ω where displacement are imposed
S_1 = part of the boundary S of Ω where tractions forces are imposed
ε = strain tensor
σ = stress tensor
t = traction field
e_m = trace of strain tensor
e_m^p = trace of plastic strain tensor
e = strain deviator
e^p = plastic strain deviator
c = cohesion
φ = internal angle of friction
θ = plastic dilatancy angle
Q = plastic potential
α^k = kinematical load factor
α^s = statical load factor
α^l = limit load factor
K_σ = set of plastically admissible stress
K_ε = set of plastically admissible strain
$\|.\|$ = Euclidian norm
ψ_K = indicator function of the set K: $\psi_K=0$ if $x\in K$ and $\psi_K(x)=+\infty$ elsewhere
grad_s = symbol of symmetric gradient
div = symbol of divergence
$\partial f(x)=\{y \mid \forall x', f(x') - f(x) \geq y.(x'-x)\}$ = subdifferential of the function f at x

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